

AD-A085 564

CALIFORNIA UNIV BERKELEY DEPT OF MECHANICAL ENGINEERING F/G 20/11  
A REMARK ON THE USE OF THE DECOMPOSITION  $F = F \text{ SUB } E \text{ F SUB } P \text{ IN--ETC(U)}$   
JAN 80 J CASEY, P M NAGHDI N00019-75-C-0148

UNCLASSIFIED

UCB/AM-80-4

NL

1 of 1  
AD-A085 564

END

DATE

FILED

7-80

DTIC

ADA 085564

DDC FILE COPY

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

12 p. 5.

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER (14) UCB/AM-80-4	2. GOVT ACCESSION NO. AD-A085564	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) A Remark on the Use of the Decomposition F = F <sub>sub</sub> in Plasticity.		5. TYPE OF REPORT & PERIOD COVERED (9) Technical Report.
7. AUTHOR(s) (10) J. Casey and P. M. Naghdi		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mechanical Engineering University of California Berkeley, CA 94720		8. CONTRACT OR GRANT NUMBER(s) (13) N00014-75-C-0148
11. CONTROLLING OFFICE NAME AND ADDRESS Structural Mechanics Branch Office of Naval Research Arlington, VA 22217		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 064-436
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) <b>LEVEL</b> (12) 14		11. REPORT DATE (11) January 1980
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited		12. NUMBER OF PAGES 12
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		15. SECURITY CLASS. (of this report) Unclassified
18. SUPPLEMENTARY NOTES		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Plasticity, nonlinear theory, invariance requirements, multiplicative decomposition, limitations, validity for anisotropic materials.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This note contains a brief discussion of the limitations involved when use is made of a multiplicative decomposition in plasticity. In particular, it is shown that if full invariance requirements are involved, then theories employing the multiplicative decomposition lose the generality attributed to them in some of the recent literature on the subject.		

DD FORM 1473  
1 JAN 73EDITION OF 1 NOV 65 IS OBSOLETE  
S/N 0102-014-6601UNCLASSIFIED  
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

400 426

80 6 13 014

OFFICE OF NAVAL RESEARCH

Contract N00014-75-C-0148

Project NR 064-436

Report No. UCB/AM-80-4

A Remark on the Use of the Decomposition

$\tilde{F} = \tilde{F}_e \tilde{F}_p$  in Plasticity

by

J. Casey and P. M. Naghdi

January 1980

Department of Mechanical Engineering

University of California

Berkeley, California

Approved for public release; distribution unlimited

# A Remark on the Use of the Decomposition $\tilde{F} = \tilde{F}_e \tilde{F}_p$ in Plasticity

by

J. Casey<sup>†</sup> and P. M. Naghdi<sup>†</sup>

## General Background

The nonlinear theory of elastic-plastic materials developed by Green and Naghdi [1,2] employs the total strain and plastic strain among its independent kinematical variables\*. Another theory by Lee [3] utilizes an intermediate stress-free configuration, together with the associated multiplicative decomposition of the deformation gradient. As pointed out by Green and Naghdi [4], Lee's development is valid only for initially isotropic materials. Other authors, among them Mandel [5] and Lubliner [6], have more recently made use of the multiplicative decomposition and have claimed that the theory of Green and Naghdi is unduly restrictive in that (see, e.g., [6, p. 165]) it is applicable only to "certain special cases of isotropy." The main purpose of this note is to show that if full invariance requirements are invoked, then the theories employing the multiplicative decomposition lose the generality attributed to them [5,6] relative to the development in [1,2]. In addition, although a complete list of references on the subject is not cited, some aspects of the present discussion will serve to clarify certain misunderstandings in the literature on plasticity involving the use of the multiplicative decomposition without satisfying full invariance requirements.

<sup>†</sup>Department of Mechanical Engineering, University of California, Berkeley.

\*The theory of elastic-plastic materials in [1,2] includes thermal effects and is developed within the framework of a thermodynamical theory. Although we confine attention to the purely mechanical aspects of the subject, the basic theory referred to here can be interpreted in the context of the isothermal theory and corresponds to a second form of the theory discussed in section 4 of [2].

Let  $X$  be a particle of an elastic-plastic body  $\mathcal{B}$  and denote by  $\underline{X}$  and  $\underline{x}$ , respectively, the positions of  $X$  in a fixed reference configuration  $\kappa_0$  and the current configuration  $\kappa$  at time  $t$ . Let  $\underline{F}$ , which for convenience we express as a function of  $X$  and  $t$ , be the deformation gradient relative to the configuration  $\kappa_0$  and recall that  $\det \underline{F} > 0$ , where  $\det$  stands for determinant. The transpose and inverse operations will be denoted by superscripts  $T$  and  $-1$ , respectively, and  $\underline{I}$  is the unit tensor.

Now it may be observed that if  $\underline{H}$  is any tensor function of  $X$  and  $t$  with  $\det \underline{H} > 0$ , then  $\underline{F} = (\underline{F} \underline{H}^{-1}) \underline{H}$  with the property that  $\det(\underline{F} \underline{H}^{-1}) > 0$ . Hence,  $\underline{F}$  can always be decomposed -- with evident nonuniqueness -- as a product in which both factors have positive determinants. Such a decomposition, namely

$$\underline{F} = \underline{F}_e \underline{F}_p, \quad (1)$$

with  $\det \underline{F}_e > 0$ ,  $\det \underline{F}_p > 0$  is used in plasticity theory. However, the use of (1) in plasticity is supplemented with further restrictions which reduce the extent of nonuniqueness but result in possible nonexistence of the decomposition. To elaborate, let  $dX$  be an arbitrary material line element of  $\mathcal{B}$  in the neighborhood of the particle  $X$  and let  $d\underline{X}$  and  $d\underline{x} = \underline{F} d\underline{X}$  be the corresponding line elements in the configurations  $\kappa_0$  and  $\kappa$ , respectively. Put  $d\underline{y} = \underline{F}_p d\underline{X}$ . Then,  $d\underline{x} = \underline{F}_e d\underline{y}$  by (1). Considering all material line elements  $dX$  at  $X$  in  $\mathcal{B}$ , we can form a local configuration from the elements  $d\underline{y}$ ; the collection of such local configurations (for all  $X$  in  $\mathcal{B}$ ) is usually referred to in the literature on plasticity as an intermediate stress-free configuration  $\bar{\kappa}$ ; in the special case of homogeneous deformations for which  $\underline{F}_e$  and  $\underline{F}_p$  are independent of  $X$ ,  $\bar{\kappa}$  will be a global configuration of  $\mathcal{B}$ . We observe that as part of the definition of  $\bar{\kappa}$ , it must be required that:

- (a) for each  $\underline{x}$ , the portion of  $\mathcal{B}$  that occupies an arbitrarily small

neighborhood of  $\underline{x}$  be reduced to a state of zero stress<sup>†</sup>; and

(b) the quantity

$$\underline{E}_{\underline{p}} = \frac{1}{2}(\underline{F}_{\underline{p}}^T \underline{F}_{\underline{p}} - \underline{I}) \quad , \quad (2)$$

called plastic strain, has the same value<sup>\*</sup> at the particle  $X$  in  $\bar{\kappa}$  and  $\kappa$ . The deformation of  $d\underline{x}$  into  $d\underline{y}$  is then interpreted as plastic and that of  $d\underline{y}$  into  $d\underline{x}$  as elastic. Let  $\underline{e}_{\underline{i}}$  and  $\underline{e}_{\underline{A}}$  be fixed orthonormal bases associated with the configurations  $\kappa$  and  $\kappa_0$ , respectively. Then, the components  $F_{iA}$  of  $\underline{F}$  referred to these bases satisfy the compatibility conditions  $\partial F_{iA} / \partial X_B = \partial F_{iB} / \partial X_A$  with respect to reference position  $\underline{X} = X_A \underline{e}_{\underline{A}}$ , while the tensors  $\underline{F}_{\underline{e}}$  and  $\underline{F}_{\underline{p}}$  in general do not satisfy any compatibility conditions; and, consequently, the configuration  $\bar{\kappa}$  cannot be mapped smoothly into  $\kappa_0$  or  $\kappa$ .

#### Issues Involved in the Use of (1)

Three main issues are involved in the use of the multiplicative decomposition (1). These are: (i) existence of a configuration such as  $\bar{\kappa}$ , (ii) uniqueness of  $\bar{\kappa}$  or equivalently of the factors  $\underline{F}_{\underline{e}}, \underline{F}_{\underline{p}}$  and (iii) the invariance requirements under superposed rigid body motions to be satisfied by  $\underline{F}_{\underline{e}}, \underline{F}_{\underline{p}}$  and their consequent effects on the constitutive equations. We discuss these issues separately.

(i) Existence. As was pointed out in [4], it is possible to reduce the stresses in a material element to zero without changing  $\underline{E}_{\underline{p}}$  if and only if the origin  $\mathcal{O}$  in stress space lies in the region  $\mathcal{S}$  bounded by the yield surface  $\partial\mathcal{S}$ . It is not always the case that  $\mathcal{O}$  belongs to  $\mathcal{S}$  and therefore if (1) is assumed, it will involve a restriction on possible constitutive equations and/or possible deformations. On the other hand, if such restrictions are not imposed, then

<sup>†</sup> The reduction to a state of zero stress is in the context of the purely mechanical theory only. The corresponding reduction in the thermodynamical theory can be discussed similarly.

<sup>\*</sup> This requirement, as already noted by Green and Naghdi [4], is implied by the usual statement that the total strain associated with the element  $d\underline{y}$  is a "plastic strain" and is equal to the plastic strain associated with the element  $d\underline{x}$ .

the decomposition (1) will not always exist.

(ii) Uniqueness. It follows from the requirement (b) that in any two intermediate stress-free configurations corresponding to the same current configuration  $\kappa$ ,  $E_p$  has the same value at the particle  $X$ . Hence, in view of (1) and (2),  $F_p$  and  $F_e$  are not unique to the extent that they are determined only to within a proper orthogonal tensor function  $^*Z$  of  $X, t$  so that  $F_e Z^T, Z F_p$  also satisfy (1) and leave the left-hand side of (2) unchanged. It then follows that the configuration  $\bar{\kappa}$  is locally determined at time  $t$  only to within a rigid displacement<sup>5</sup>.

(iii) Invariance Requirements. First we recall that in response to certain remarks made by Lee [3], Green and Naghdi [4] studied the possibility of accommodating the decomposition (1) within the framework of their general thermodynamical theory [1,2] in which, in addition to temperature  $\theta$  and work-hardening parameter  $\kappa$ , the kinematical variables were the total strain  $E = \frac{1}{2}(F^T F - I)$  and the plastic strain  $E_p$  introduced as a primitive variable. In [1,2]  $E_p$  and  $\kappa$  were assumed to be unaltered under superposed rigid body motions. It was established in [4] that by assuming the decomposition (1) and making the identification between the primitive quantity  $E_p$  in [1,2] and the defined quantity  $E_p$  in (2), that a theory utilizing the variables  $F_e$  and  $F_p$  could be derived from that of Green and Naghdi [1,2].

With reference to the invariance requirements, we recall that physical considerations demand that certain fields and functions entering the theory be indifferent<sup>\*\*</sup> to any transformation which takes the present configuration  $\kappa$

\* There is a dependency on  $X$  since the stress-free configuration is local.

<sup>5</sup> At this stage of our discussion, it cannot be said whether the requirement (a) can reduce this lack of uniqueness. We return to this later; see the end of the paragraph containing (3).

\*\* We use the term indifferent for brevity to mean unaltered or unaltered apart from orientation as defined in [7]. The notations  $F^*, E^*$ , etc., here are in line with those in [7] and correspond to  $\tilde{F}^*, \tilde{E}^*$ , etc., in [4].

of a body rigidly into a configuration  $\kappa^+$ . Since  $\bar{\kappa}$  is locally just another configuration, then by the same physical reasoning, it was assumed in [4] that these fields and functions are also indifferent to a transformation that independently replaces the intermediate configuration  $\bar{\kappa}$  by a configuration  $\bar{\kappa}^+$  related to  $\bar{\kappa}$  through a superposed rigid body motion. Let  $\kappa \rightarrow \kappa^+$  and  $\bar{\kappa} \rightarrow \bar{\kappa}^+$  by independent superposed rigid body motions. Then, we have the transformations  $F \rightarrow F^+$ ,  $F_e \rightarrow F_e^+$  and  $F_p \rightarrow F_p^+$  with

$$\tilde{F}^+ = \tilde{Q}(t)\tilde{F} = \tilde{F}\tilde{F}_p^+ \quad , \quad \tilde{F}_e^+ = \tilde{Q}(t)\tilde{F}_e\tilde{Q}^T(t) \quad , \quad \tilde{F}_p^+ = \tilde{Q}(t)\tilde{F}_p \quad , \quad (3)$$

where  $\tilde{Q}(t)$  and  $\bar{Q}(t)$  are proper orthogonal tensor-valued functions of time only corresponding, respectively, to the arbitrary rigid body rotations in the motions through which  $\kappa \rightarrow \kappa^+$  and  $\bar{\kappa} \rightarrow \bar{\kappa}^+$ . Clearly,  $E_p$  in (2) and  $E$  will remain unaltered under the above transformations and the work-hardening parameter  $\kappa$  is assumed to also remain unaltered. Furthermore we assume that the stress tensor which appears in the constitutive discussion, namely the symmetric Piola-Kirchhoff stress  $\tilde{S}$  for both configurations  $\bar{\kappa}$  and  $\kappa$ , remains unaltered under the transformations  $\kappa \rightarrow \kappa^+$ ,  $\bar{\kappa} \rightarrow \bar{\kappa}^+$ . It is then clear that  $\bar{\kappa}$  satisfies requirements (a) and (b) above if and only if  $\bar{\kappa}^+$  does, i.e.,  $\bar{\kappa}^+$  is an intermediate stress-free configuration if and only if  $\bar{\kappa}$  is. In particular, since we can now conclude that  $\tilde{S} = 0$  in  $\bar{\kappa}^+$  if  $\tilde{S} = 0$  in  $\bar{\kappa}$ , it follows that requirement (a) does not further reduce the lack of uniqueness mentioned under (ii).

The invariance requirements (3) were observed by Green and Naghdi [4]<sup>§</sup> and were later used by Naghdi and Trapp [9] in effecting an essential reduction in the form of the strain energy response function. In contrast to the full invariance requirements, those adopted by many authors correspond to (3) but with  $\bar{Q}(t) = I$ . In some cases, for example [3], erroneous results were avoided

<sup>§</sup> Invariance requirements of the form (3) were also adopted independently by Sidoroff [8], although he appeals to the principle of material frame-indifference.



because the analysis was restricted to isotropic materials. However, Mandel [5] introduces the idea of an<sup>†</sup> "isoclinic" stress-free intermediate configuration  $\hat{\kappa}$ , i.e., one that has a fixed orientation relative to a set of axes in space, and adopts invariance requirements corresponding to (3) with  $\bar{Q}(t) = I$ . The notion of a fixed orientation used in [5] is itself not an invariant one. For Mandel's scheme to have any physical relevance, his results must be indifferent to the choice of fixed orientation. This leads one to demand that the full invariance requirements<sup>\*</sup> (3) be satisfied. We discuss this further below, but note here that Mandel's scheme is adopted by Lubliner [6] who repeats the criticisms of [1,4] stated in [5].

We have already indicated that the definition of a stress-free configuration (involving the requirements (a) and (b) noted earlier) determines  $\bar{\kappa}$  only to within a rigid displacement at time  $t$ . In this connection, it is perhaps natural to ask if by introducing a further assumption one could choose a unique  $\bar{\kappa}$  from among all possible intermediate stress-free configurations and thereby obtain a unique choice for  $F_p$ . In examining this possibility, one is immediately led to conclude that unless a nonuniqueness of rotation  $\bar{Q}$  remains in  $\bar{\kappa}$ , the full invariance requirements (3) will not be satisfied. To elaborate, consider for example a possible additional assumption that  $F_e$  be symmetric positive definite. Then, application of (3)<sub>2</sub> shows that  $F_e^+$  is not symmetric positive definite unless  $\bar{Q}(t)$  is set equal to  $Q(t)$  in (3). But, such a stipulation on the invariance requirements (although it may be specified mathematically) is unduly restrictive on physical grounds. Similarly, an

<sup>†</sup>Actually, Mandel [5, p. 728] employs the terminology "configurations isoclines."

<sup>\*</sup>It is clear that any chosen isoclinic configuration  $\hat{\kappa}$  at time  $t$  may be regarded as corresponding to  $\bar{Q}(t) = I$ . A different choice of orientation will then result in a different  $\bar{Q}(t)$ . If this choice is to be arbitrary, then  $\bar{Q}(t)$  must be arbitrary also.

assumption that  $\underline{F}_p$  be symmetric positive definite is not an invariant idea since  $\underline{F}_p^+$  will not be symmetric positive definite. It should be kept in mind that the invariance requirements (3) embody the idea that at time  $t$  all intermediate stress-free configurations differing from one another by a rigid displacement are physically indistinguishable and there are no physical grounds for choosing one of them rather than another. However, while  $\underline{\kappa}$  and hence  $\underline{F}_p$  cannot be chosen uniquely, it is important to note that  $\underline{E}_p$  can be chosen uniquely, for example through the definition (2).

#### Implication of Invariance Requirements Stated Under (iii)

In what follows we shall need to have available some results from [1,2,4]. Interpreting the isothermal case of the theory in [1,2] as corresponding to the purely mechanical theory we obtain

$$\psi = \hat{\psi}(\underline{E}, \underline{E}_p, \kappa) \quad , \quad \underline{S} = \rho_0 \frac{\partial \hat{\psi}}{\partial \underline{E}} \quad , \quad (4)$$

as properly invariant constitutive equations for the strain energy per unit mass and the symmetric Piola-Kirchhoff stress tensor  $\underline{S}$ , where  $\rho_0$  is the mass density in the configuration  $\underline{\kappa}_0$ . It is understood that the response function  $\hat{\psi}$  in (4) is expressed as a symmetric function of  $\underline{E}$ . In addition, for fixed values of  $\underline{E}_p$  and  $\kappa$ , the yield surface  $\partial \mathcal{S}$  in stress space is given in invariant form by

$$\phi(\underline{S}, \underline{E}_p, \kappa) = 0 \quad . \quad (5)$$

In the theory of Green and Naghdi [1,2],  $\underline{E}_p$  is a primitive kinematical quantity and no kinematical relation between  $\underline{E}$  and  $\underline{E}_p$  is assumed. The limitations concerning existence discussed under (i) above do not arise in this general framework; and, if only for this reason, it seems to be preferable in a general theory of plasticity to employ  $\underline{E}$  and  $\underline{E}_p$  rather than  $\underline{F}_e$  and  $\underline{F}_p$ . Again for the same reason,

it seems preferable when using  $\underline{\tilde{E}}$  and  $\underline{\tilde{E}}_p$  not to introduce stress-free configurations as part of the general theory\*. As soon as the identification between the primitive  $\underline{\tilde{E}}_p$  of [1,2] and the defined quantity  $\underline{\tilde{E}}_p$  in (2) is made, the theory of [1,2] loses some of its generality and the discussion (i) of existence becomes relevant. In the remainder of this note we assume that this identification has been made.

We now recall polar decompositions of the invertible tensors  $\underline{\tilde{F}}, \underline{\tilde{F}}_e$  and  $\underline{\tilde{F}}_p$  and define deformation tensors  $\underline{\tilde{C}}, \underline{\tilde{C}}_e$  and  $\underline{\tilde{C}}_p$  as follows:

$$\underline{\tilde{F}} = \underline{\tilde{R}} \underline{\tilde{M}} \quad , \quad \underline{\tilde{C}} = \underline{\tilde{F}}^T \underline{\tilde{F}} = \underline{\tilde{M}}^2 \quad , \quad (6)$$

$$\underline{\tilde{F}}_e = \underline{\tilde{R}}_e \underline{\tilde{M}}_e \quad , \quad \underline{\tilde{C}}_e = \underline{\tilde{M}}_e^2 \quad ,$$

and

$$\underline{\tilde{F}}_p = \underline{\tilde{R}}_p \underline{\tilde{M}}_p \quad , \quad \underline{\tilde{C}}_p = \underline{\tilde{M}}_p^2 = 2\underline{\tilde{E}}_p + \underline{\tilde{I}} \quad , \quad (7)$$

where  $\underline{\tilde{R}}, \underline{\tilde{R}}_e, \underline{\tilde{R}}_p$  are proper orthogonal tensors and  $\underline{\tilde{M}}, \underline{\tilde{M}}_e, \underline{\tilde{M}}_p$  are symmetric positive definite tensors. We note that in view of (1), (6) and (7),  $\underline{\tilde{C}}$  may be expressed as

$$\underline{\tilde{C}} = \underline{\tilde{M}}_p \underline{\tilde{R}}_p^T \underline{\tilde{C}}_e \underline{\tilde{R}}_p \underline{\tilde{M}}_p \quad . \quad (8)$$

When  $\underline{\tilde{\kappa}} \rightarrow \underline{\tilde{\kappa}}^+$ ,  $\underline{\tilde{\kappa}} \rightarrow \underline{\tilde{\kappa}}^+$ , then  $\underline{\tilde{R}} \rightarrow \underline{\tilde{R}}^+$ ,  $\underline{\tilde{M}} \rightarrow \underline{\tilde{M}}^+$ , etc., and it can be deduced from (3), (6) and (7) that

$$\begin{aligned} \underline{\tilde{R}}^+ &= \underline{\tilde{Q}}(t) \underline{\tilde{R}} \quad , \quad \underline{\tilde{M}}^+ = \underline{\tilde{M}} \quad , \quad \underline{\tilde{C}}^+ = \underline{\tilde{C}} \quad , \quad \underline{\tilde{E}}^+ = \underline{\tilde{E}} \quad , \\ \underline{\tilde{R}}_e^+ &= \underline{\tilde{Q}}(t) \underline{\tilde{R}}_e \underline{\tilde{Q}}^T(t) \quad , \quad \underline{\tilde{M}}_e^+ = \underline{\tilde{Q}}(t) \underline{\tilde{M}}_e \underline{\tilde{Q}}^T(t) \quad , \quad \underline{\tilde{C}}_e^+ = \underline{\tilde{Q}}(t) \underline{\tilde{C}}_e \underline{\tilde{Q}}^T(t) \quad , \\ \underline{\tilde{R}}_p^+ &= \underline{\tilde{Q}}(t) \underline{\tilde{R}}_p \quad , \quad \underline{\tilde{M}}_p^+ = \underline{\tilde{M}}_p \quad , \quad \underline{\tilde{C}}_p^+ = \underline{\tilde{C}}_p \quad , \quad \underline{\tilde{E}}_p^+ = \underline{\tilde{E}}_p \quad . \end{aligned} \quad (9)$$

\* In special cases, of course, it may be desirable for purposes of interpretation or experimental identification to make use of such stress-free configurations in order to identify  $\underline{\tilde{E}}_p$  by the form (2). Another way of identifying plastic strain is through the use of an assumption which would require that  $\underline{\tilde{E}}_p$  reduce to  $\underline{\tilde{E}}$  when  $\underline{\tilde{S}} = \underline{\tilde{0}}$  (see property 3 on p. 122 of [2]).

Once the decomposition (1) is admitted, the strain energy  $\psi$  may be expressed in the equivalent forms

$$\psi = \psi_1(\underline{F}_e, \underline{F}_p, \kappa) = \psi_2(\underline{F}, \underline{F}_p, \kappa) \quad (10)$$

Since  $\psi$  must remain unaltered under the transformations  $\kappa \rightarrow \kappa^+$ ,  $\bar{\kappa} \rightarrow \bar{\kappa}^+$ , then considering first the function  $\psi_1$  in (10)<sub>1</sub> we obtain

$$\psi = \psi^+ = \psi_1(\underline{F}_e^+, \underline{F}_p^+, \kappa^+) = \psi_1(\underline{Q}(t) \underline{F}_e \underline{Q}^T(t), \underline{Q}(t) \underline{F}_p, \kappa) \quad (11)$$

for arbitrary proper orthogonal  $\underline{Q}(t), \bar{\underline{Q}}(t)$ , where (3)<sub>2,3</sub> have been used.

Recalling (6)<sub>3</sub> and (7)<sub>1</sub> we choose  $\bar{\underline{Q}}(t) = \underline{R}_p^T$  and  $\underline{Q}(t) = \underline{R}_{pe}^T \underline{R}_e^T$  in (11) so that

$$\psi = \psi_1(\underline{R}_{pe}^T \underline{M} \underline{R}_p, \underline{M}_p, \kappa) \quad (12)$$

and we note the presence of  $\underline{R}_p$  in the arguments of (12). We have shown that a necessary condition for the satisfaction of invariance requirements is that  $\psi_1$  in (10)<sub>1</sub> depend on  $\underline{F}_e, \underline{F}_p, \kappa$  only through the arguments appearing in (12). It is readily seen, with the help of (9), that taking  $\psi_1$  in the form (12) is also sufficient for the satisfaction of invariance requirements. Observing the relations  $\underline{R}_{pe}^T \underline{M} \underline{R}_p = (\underline{R}_{pe}^T \underline{C} \underline{R}_p)^{\frac{1}{2}}$  and  $\underline{M}_p = \underline{C}_p^{\frac{1}{2}}$ , we can express  $\psi$  as a properly invariant function of  $\underline{R}_{pe}^T \underline{C} \underline{R}_p, \underline{C}_p, \kappa$ . This was the form used in section 4 of [4]. Considering now the function  $\psi_2$  in (10)<sub>2</sub>, it can be shown by a similar argument that a necessary and sufficient condition for the satisfaction of invariance requirements is that  $\psi_2$  can depend on  $\underline{F}, \underline{F}_p, \kappa$  only in the forms

$$\psi = \psi_2(\underline{M}, \underline{M}_p, \kappa) = \psi_3(\underline{C}, \underline{C}_p, \kappa) = \hat{\psi}(\underline{E}, \underline{E}_p, \kappa) \quad (13)$$

the last of which is that employed in [1,2,4,9]. Indeed, in view of (8), the reduced forms of (12) and (13) are equivalent.

We now return to Mandel's development [5] and introduce the notations  $\hat{\underline{F}}_p, \hat{\underline{R}}_p, \hat{\underline{M}}_p$  for the values of  $\underline{F}_p, \underline{R}_p, \underline{M}_p$  associated with his "isoclinic" stress-free configuration  $\hat{\kappa}$ , as well as  $\hat{\underline{\pi}} = \hat{\underline{F}}_p \hat{\underline{S}} \hat{\underline{F}}_p^T / \det \hat{\underline{F}}_p$ . A typical result in Mandel's

development is an equation of the form<sup>†</sup>

$$\hat{\phi}(\hat{\pi}) = 0 \quad (14)$$

for a yield surface in stress space. Applying the invariance requirements (3) and assuming  $\hat{\phi}$  to be invariant, we obtain

$$\hat{\phi}(\hat{\pi}) = \hat{\phi}(\hat{\pi}^+) = \hat{\phi}(\hat{Q}(t) \hat{F}_{\sim p} \hat{S} \hat{F}_{\sim p}^T \hat{Q}^T / \det \hat{F}_{\sim p}) \quad (15)$$

for arbitrary proper orthogonal  $\hat{Q}(t)$ . With the help of (15), the polar decomposition  $\hat{F}_{\sim p} = \hat{R}_{\sim p} \hat{M}_{\sim p}$ , the fact that  $\hat{M}_{\sim p} = \hat{M}_{\sim p}$  by virtue of requirement (b) and choosing  $\hat{Q}(t) = \hat{R}_{\sim p}^T$ , (14) reduces to

$$\hat{\phi}(\hat{M}_{\sim p} \hat{S} \hat{M}_{\sim p}) = 0 \quad (16)$$

Thus,  $\hat{\phi}$  can depend on the argument  $\hat{\pi}$  only through  $\hat{S}$  and  $\hat{M}_{\sim p}$ . Clearly, the left-hand side of (16) can be written as a different function  $\hat{\phi}(\hat{S}, \hat{E}_{\sim p})$ . Hence, apart from the work-hardening parameter  $\kappa$  not included in (14), the form (16) of the yield surface is equivalent to (5) which is that used by Green and Naghdi [1,2]. Parallel arguments apply to other relevant equations in [5] and it should now be clear that the criticism of [1,4] by Mandel and others who have adopted his scheme is unjustified.

It should be emphasized that in deducing (12), (13) and (16) no assumptions were made concerning material symmetry and consequently these equations are valid for a material which is anisotropic in its reference configuration. Some authors, for example Mandel [5], regard equations such as (5) and (13) to be valid only for special materials which are "isotropic in the intermediate configuration." However, we have just seen that the invariance requirements (3) imply that constitutive equations such as (10)<sub>2</sub> always satisfy an equation of the form (13).

Finally a comment must be made about a paper by Silhavy [10]. In the

---

<sup>†</sup>The notation  $\hat{\phi}$  in (14) corresponds to  $\hat{f}$  in (8.4) of [5] and we have suppressed Mandel's variables  $T, \hat{Q}$ , since they do not affect the present discussion. It is important to note that we would still employ the full invariance requirements (3) even if these variables were included. Our  $\hat{F}_{\sim p}, \hat{R}_{\sim p}, \hat{M}_{\sim p}, \hat{S}$  correspond, respectively, to  $\hat{P}, \hat{Q}, \hat{L}, \pi_0$  of [5] and the work-hardening parameter  $\kappa$  is not explicitly exhibited in (14).

context of a functional type theory, he has attempted to prove that the appropriate transformation law for  $F_{\tilde{p}}$  is  $F_{\tilde{p}} \rightarrow F_{\tilde{p}}$  (or  $p \rightarrow p$  in the notation of [10]), i.e.,  $\bar{Q}(t) = \underline{I}$  in (3). However, his main proposition (Proposition 4 in [10]) states that a certain set  $P_{qf}(\pi_o, \tilde{N})$  is equal to a set  $P_f(\pi_o, \tilde{N})$ . As Silhavy himself points out, there may be more than one element in the set  $P_{qf}(\pi_o, \tilde{N})$ . Therefore, it cannot be deduced from Silhavy's Proposition 4 that  $F_{\tilde{p}} \rightarrow F_{\tilde{p}}$  under superposed rigid body motions\*.

Acknowledgment. The results reported here were obtained in the course of research supported by the U.S. Office of Naval Research under Contract N00014-75-C-0148, Project NR 064-436, with the University of California, Berkeley.

---

\* For example, in plane geometry a rotation of the unit circle  $C$  through an angle of 45 degrees, say, maps  $C$  into itself but it certainly cannot be argued that each point of  $C$  is mapped into itself.

# References

1. Green, A. E., and Naghdi, P. M., "A General Theory of an Elastic-Plastic Continuum," Archive for Rational Mechanics and Analysis, Vol. 18, 1965, pp. 251-281.
2. Green, A. E., and Naghdi, P. M., "A Thermodynamic Development of Elastic-Plastic Continua," Proceedings of the IUTAM Symposium on Irreversible Aspects of Continuum Mechanics and Transfer of Physical Characteristics in Moving Fluids, Springer-Verlag, 1966, pp. 117-131.
3. Lee, E. H., "Elastic-Plastic Deformation at Finite Strains," Journal of Applied Mechanics, Vol. 36, Trans ASME, Vol. 91, Series E, 1969, pp. 1-6.
4. Green, A. E., and Naghdi, P. M., "Some Remarks on Elastic-Plastic Deformation at Finite Strain," International Journal of Engineering Science, Vol. 9, 1971, pp. 1219-1229.
5. Mandel, J., "Equations Constitutives et Directeurs dans les Milieux Plastiques et Viscoplastiques," International Journal of Solids and Structures, Vol. 9, 1973, pp. 725-740.
6. Lubliner, J., "A Thermodynamic Yield Criterion in Viscoplasticity," Acta Mechanica, Vol. 30, 1978, pp. 165-174.
7. Green, A. E., and Naghdi, P. M., "A Note on Invariance Under Superposed Rigid Body Motions," Journal of Elasticity, Vol. 9, 1979, pp. 1-8.
8. Sidoroff, F., "Quelques Réflexions sur le Principe d'Indifférence Matérielle pour un Milieu ayant un Etat Relâché," Compte Rendus de l'Académie des Sciences, Paris, Vol. 271, Série A, 1970, pp. 1026-1029.
9. Naghdi, P. M., and Trapp, J. A., "On Finite Elastic-Plastic Deformation of Metals," Journal of Applied Mechanics, Vol. 41, 1974, pp. 254-260.
10. Silhavy, M., "On Transformation Laws for Plastic Deformations of Materials with Elastic Range," Archive for Rational Mechanics and Analysis, Vol. 63, 1977, pp. 169-182.

Accession For	
NTIS GRA&I	<input checked="checked" type="checkbox"/>
DDC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
Distribution/	
Availability Codes	
Dist	Avail and/or special
A	